

# TENSOR CATEGORIES ATTACHED TO EXCEPTIONAL CELLS IN WEYL GROUPS

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ABSTRACT. Using truncated convolution of perverse sheaves on a flag variety Lusztig associated a monoidal category to a two sided cell in the Weyl group. We identify this category in the case which was not decided previously.

## 1. INTRODUCTION

Let  $W$  be a finite irreducible crystallographic Coxeter group and let  $\text{Irr}(W)$  be the set of isomorphism classes of irreducible representations of  $W$  over  $\mathbb{Q}$ . It is well known that two irreducible representations of dimension 512 for  $W$  of type  $E_7$  and four irreducible representations of dimension 4096 for  $W$  of type  $E_8$  behave differently from other elements of  $\sqcup_W \text{Irr}(W)$  in many ways, see e.g. [Cu], [BL], [L1, Theorem 4.23] (see [L1, p. 109] for the definition of function  $\Delta$ ). For this reason the representations above are called *exceptional*. In this note we give one more example of unusual behavior of exceptional representations or rather of associated geometric objects.

Recall that the group  $W$  is partitioned into *two sided cells*,  $W = \sqcup \mathbf{c}$ , see e.g. [L1, p. 137]. We say that a two sided cell  $\mathbf{c}$  is *exceptional* if the corresponding *family* (see [L1, 5.15, 5.25]) consists of exceptional representations of  $W$ . Thus there are just three exceptional two sided cells, one for  $W$  of type  $E_7$  and two for  $W$  of type  $E_8$ , see [L1, Chapter 4].

We fix an algebraically closed field  $\mathbb{K}$  of characteristic zero. Let  $G$  be a simple algebraic group over  $\mathbb{C}$  with the Weyl group  $W$ . Using truncated convolution of perverse  $\mathbb{K}$ -sheaves (in the classical topology) on the flag variety of  $G$ , Lusztig associated to each two sided cell  $\mathbf{c} \subset W$  a semisimple monoidal category  $\mathcal{P}^{\mathbf{c}}$  (over  $\mathbb{K}$ ), see [L4] and §2.3 below. Moreover, Lusztig conjectured that there is a tensor equivalence  $\mathcal{P}^{\mathbf{c}} \simeq \text{Coh}_{A(\mathbf{c})}(Y \times Y)$  where  $A(\mathbf{c})$  is a finite group associated with family corresponding to  $\mathbf{c}$  in [L1, Chapter 4],  $Y$  is a finite  $A(\mathbf{c})$ -set and  $\text{Coh}_{A(\mathbf{c})}(Y \times Y)$  is the category of  $A(\mathbf{c})$ -equivariant sheaves on the set  $Y \times Y$  with convolution as a tensor product and a natural associativity constraint, see [L4, §3.2]. This conjecture was verified in [BFO1] for all non-exceptional two sided cells.

Let  $\mathbf{c}$  be an exceptional cell. Then  $A(\mathbf{c}) = \mathbb{Z}/2\mathbb{Z}$  is the cyclic group of order 2. The Lusztig's conjecture from [L4] predicts that we have a tensor equivalence  $\mathcal{P}^{\mathbf{c}} \simeq \text{Coh}_{A(\mathbf{c})}(Y \times Y)$  where  $Y$  is a finite set with *free*  $A(\mathbf{c}) = \mathbb{Z}/2\mathbb{Z}$ -action (the set  $Y$  should be of cardinality 1024 if  $W$  is of type  $E_7$  and of cardinality 8192 if  $W$  is of type  $E_8$ ). Our main goal is to show that one needs to change the associativity constraint a little bit in order to make this statement correct.

Let  $Y'$  be a finite set of cardinality 512 if  $W$  is of type  $E_7$  and of cardinality 4096 if  $W$  is of type  $E_8$  (one can identify  $Y'$  with the set of  $\mathbb{Z}/2\mathbb{Z}$ -orbits on the set  $Y$ ). Let  $\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}$  be the monoidal category of finite dimensional  $\mathbb{Z}/2\mathbb{Z}$ -graded spaces. Then there is a tensor equivalence  $\text{Coh}_{\mathbb{Z}/2\mathbb{Z}}(Y \times Y) \simeq \text{Vec}_{\mathbb{Z}/2\mathbb{Z}} \boxtimes \text{Coh}(Y' \times Y')$ . The category  $\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}$  has two simple objects, the unit object  $\mathbf{1}$  and one more simple object  $\delta$ . Let  $\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}^\omega$  be the same as category  $\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}$  but with modified associativity constraint: for simple objects  $X, Y, Z$  the associativity constraint  $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$  is the same as in the category  $\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}$  if at least one of  $X, Y, Z$  is isomorphic to  $\mathbf{1}$  and the associativity constraint  $(\delta \otimes \delta) \otimes \delta \simeq \delta \otimes (\delta \otimes \delta)$  in  $\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}^\omega$  differs by sign from the one in  $\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}$ . Here is our main result:

**Theorem 1.1.** *For an exceptional two sided cell  $\mathbf{c}$  there is a tensor equivalence  $\mathcal{P}^{\mathbf{c}} \simeq \text{Vec}_{\mathbb{Z}/2\mathbb{Z}}^\omega \boxtimes \text{Coh}(Y' \times Y')$ .*

**Remark 1.2.** (i) It will be clear from the proof that the category  $\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}^\omega \boxtimes \text{Coh}(Y' \times Y')$  is not equivalent to  $\text{Vec}_{\mathbb{Z}/2\mathbb{Z}} \boxtimes \text{Coh}(Y' \times Y')$ .

(ii) Using the methods from [BFO1] one can show that for an exceptional two sided cell  $\mathbf{c}$  one has either  $\mathcal{P}^{\mathbf{c}} \simeq \text{Vec}_{\mathbb{Z}/2\mathbb{Z}} \boxtimes \text{Coh}(Y' \times Y')$  or  $\mathcal{P}^{\mathbf{c}} \simeq \text{Vec}_{\mathbb{Z}/2\mathbb{Z}}^\omega \boxtimes \text{Coh}(Y' \times Y')$ . Thus our main result is just a computation of the sign in the associativity constraint.

(iii) Lusztig's construction of the category  $\mathcal{P}^{\mathbf{c}}$  and our arguments extend with trivial changes to the settings of  $D$ -modules ( $G$  is defined over an algebraically closed field of characteristic zero) or  $\ell$ -adic sheaves ( $G$  defined over an algebraically closed field in which  $\ell$  is invertible).

(iv) There is an alternative definition of the category  $\mathcal{P}^{\mathbf{c}}$  based on idea of A. Joseph. Namely, it is proved in [BFO2, Corollary 4.5(b)] that ( $D$ -module counterpart of) the category  $\mathcal{P}^{\mathbf{c}}$  is tensor equivalent to certain subquotient of the category of Harish-Chandra bimodules with tensor product coming from the usual tensor product of bimodules.

(v) The arguments in the proof of Theorem 1.1 extend to the case of monodromic sheaves on the flag variety as in [BFO1, §5]. Thus we get a proof of statement in [BFO1, Remark 5.7].

A direct computation of the associativity constraint in the category  $\mathcal{P}^{\mathbf{c}}$  seems to be difficult. Thus our proof of Theorem 1.1 is indirect. Our main tool is the theory of (unipotent) *character sheaves* developed by Lusztig [L3]. We use a *commutator functor* (see [BFO1, §6]) from  $\mathcal{P}^{\mathbf{c}}$  to the category of sheaves on the group  $G$  (values of this functor are direct sums of character sheaves on  $G$ ). Our main observation is that a commutator functor carries a canonical automorphism and keeping track of the order of this automorphism is sufficient in order to distinguish between categories  $\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}$  and  $\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}^\omega$ .

## 2. PROOFS

**2.1. Commutator functors.** Let  $\mathcal{C}$  be a monoidal category with associativity isomorphism  $a$  and let  $\mathcal{A}$  be a category.

**Definition 2.1.** ([BFO1, §6]) A *commutator functor*  $F : \mathcal{C} \rightarrow \mathcal{A}$  is a functor endowed with a natural isomorphism  $u_{X,Y} : F(X \otimes Y) \simeq F(Y \otimes X)$  such that the

following diagram commutes:

$$\begin{array}{ccccc}
 & & F(X \otimes (Y \otimes Z)) & & \\
 & \nearrow^{F(a_{X,Y,Z})} & & \nwarrow^{u_{X,Y \otimes Z}} & \\
 F((X \otimes Y) \otimes Z) & & & & F((Y \otimes Z) \otimes X) \\
 \downarrow^{u_{X \otimes Y, Z}} & & & & \downarrow^{F(a_{Y,Z,X})} \\
 F(Z \otimes (X \otimes Y)) & & & & F(Y \otimes (Z \otimes X)) \\
 & \nwarrow^{F(a_{Z,X,Y}^{-1})} & & \nearrow^{u_{Z \otimes X, Y}} & \\
 & & F((Z \otimes X) \otimes Y) & & 
 \end{array}$$

**Remark 2.2.** It follows immediately from definition that  $u_{X,1} : F(X) \rightarrow F(X)$  is an idempotent and hence an identity map.

**Definition 2.3.** The map  $u_{1,X} : F(X) \rightarrow F(X)$  is called *canonical automorphism* of the commutator functor  $F$ .

**Remark 2.4.** It follows from definition that  $u_{Y,X} \circ u_{X,Y} = u_{1,X \otimes Y}$ .

Recall that a *central functor*  $G : \mathcal{A} \rightarrow \mathcal{C}$  is a functor endowed with functorial isomorphism  $v_{A,X} : G(A) \otimes X \simeq X \otimes G(A)$  such that the following diagram commutes:

$$\begin{array}{ccccc}
 & & G(A) \otimes (X \otimes Y) & & \\
 & \nearrow^{a_{G(A),X,Y}} & & \nwarrow^{v_{A,X \otimes Y}} & \\
 (G(A) \otimes X) \otimes Y & & & & (X \otimes Y) \otimes G(A) \\
 \downarrow^{v_{A,X} \otimes \text{id}_Y} & & & & \downarrow^{a_{X,Y,G(A)}} \\
 (X \otimes G(A)) \otimes Y & & & & X \otimes (Y \otimes G(A)) \\
 & \nwarrow^{a_{X,G(A),Y}} & & \nearrow^{\text{id}_X \otimes v_{A,Y}} & \\
 & & X \otimes (G(A) \otimes Y) & & 
 \end{array}$$

Equivalently, the structure of central functor on  $G$  is the same as factorization  $\mathcal{A} \rightarrow \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$  where  $\mathcal{Z}(\mathcal{C})$  is the Drinfeld center of  $\mathcal{C}$  and  $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$  is the forgetful functor, see e.g. [ENO, §2.3].

**Proposition 2.5.** ([BFO1, §6]) *Assume that the category  $\mathcal{C}$  is rigid and has a pivotal structure (that is an isomorphism of tensor functors  $\text{Id} \rightarrow **$ ). Let  $(G, F)$  be an adjoint pair of functors between  $\mathcal{C}$  and  $\mathcal{A}$ . Then the structures of commutator functor on  $F$  are in natural bijection with structures of central functor on  $G$ .*

*Proof.* Assume that  $G$  has a central structure. Then

$$\begin{aligned}
 \text{Hom}(?, F(X \otimes Y)) &= \text{Hom}(G(?), X \otimes Y) = \text{Hom}(G(?) \otimes *Y, X) \\
 &= \text{Hom}(*Y \otimes G(?), X) = \text{Hom}(G(?), **Y \otimes X) = \text{Hom}(?, F(Y \otimes X))
 \end{aligned}$$

and it is straightforward to check that the resulting isomorphism  $F(X \otimes Y) \simeq F(Y \otimes X)$  satisfies the definition of commutator functor.

Conversely, assume that  $F$  has a commutator structure. Then

$$\begin{aligned} \mathrm{Hom}(G(A) \otimes X, ?) &= \mathrm{Hom}(G(A), ? \otimes X^*) = \mathrm{Hom}(A, F(? \otimes X^*)) \\ &= \mathrm{Hom}(A, F(X^* \otimes ?)) = \mathrm{Hom}(G(A), X^* \otimes ?) = \mathrm{Hom}(X \otimes G(A), ?) \end{aligned}$$

and it is straightforward to check that the resulting isomorphism  $G(A) \otimes X \simeq X \otimes G(A)$  satisfies the definition of central functor.

Finally, one verifies that the two constructions above are mutually inverse.  $\square$

Let  $\mathcal{C}$  be a pivotal category. Assume that the forgetful functor  $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{C}$  has a right adjoint functor  $\mathrm{Ind} : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$ . It follows from Proposition 2.5 that the functor  $\mathrm{Ind}$  has a canonical structure of commutator functor. Clearly, for any functor  $\mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{A}$  the composition  $\mathcal{C} \xrightarrow{\mathrm{Ind}} \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{A}$  has a structure of commutator functor. Moreover, we have the following universal property:

**Corollary 2.6.** *Let  $F : \mathcal{C} \rightarrow \mathcal{A}$  be a commutator functor such that left adjoint of  $F$  exists. Then  $F$  factorizes as  $\mathcal{C} \xrightarrow{\mathrm{Ind}} \mathcal{Z}(\mathcal{C}) \rightarrow \mathcal{A}$ .  $\square$*

Let  $\mathcal{D}$  be a ribbon category, for example Drinfeld center  $\mathcal{Z}(\mathcal{C})$  of a spherical category  $\mathcal{C}$ . Recall that there is a canonical automorphism  $\theta$  of the identity functor called *twist* defined as following composition:

$$X \rightarrow **X \otimes *X \otimes X \rightarrow **X \otimes X \otimes *X \rightarrow **X = X.$$

One can rewrite this definition in terms of functor represented by  $X$  as follows:

$$\mathrm{Hom}(?, X) = \mathrm{Hom}(? \otimes *X, \mathbf{1}) = \mathrm{Hom}(*X \otimes ?, \mathbf{1}) = \mathrm{Hom}(?, **X).$$

Thus we have the following

**Corollary 2.7.** *The canonical automorphism of the commutator functor  $\mathrm{Ind} : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$  equals the twist automorphism.  $\square$*

**Example 2.8.** (a) Let  $\mathcal{C} = \mathrm{Vec}_{\mathbb{Z}/2\mathbb{Z}}$ . It is well known that the category  $\mathcal{C}$  is spherical (in two different ways) and the twist of any simple object of  $\mathcal{Z}(\mathcal{C})$  is  $\pm 1$  (for any choice of the spherical structure). It follows that for any commutator functor  $F : \mathcal{C} \rightarrow \mathcal{A}$  such that left adjoint of  $F$  exists the square of the canonical automorphism is the identity map.

(b) Let  $\mathcal{C} = \mathrm{Vec}_{\mathbb{Z}/2\mathbb{Z}}^\omega$ . The category  $\mathcal{C}$  has two spherical structures and the twists of simple objects of  $\mathcal{Z}(\mathcal{C})$  are 1 and  $\pm\sqrt{-1}$ . Thus there exists a commutator functor  $F : \mathcal{C} \rightarrow \mathcal{A}$  such that left adjoint of  $F$  exists and the square of the canonical automorphism is not the identity map, for example  $F = \mathrm{Ind}$ .

**2.2. Commutator functor  $\Gamma$  and its canonical automorphism.** Let  $G$  be a simple algebraic group over an algebraically closed field and let  $B \subset G$  be a Borel subgroup. The group  $B \times B$  acts on  $G$  via left and right translations. We consider the subcategories  $\mathcal{C}_{B \times B}$  and  $\mathcal{C}_G$  of the bounded derived category of constructible sheaves on  $G$  consisting of complexes that are isomorphic to a finite direct sum of shifts of simple,  $B \times B$ -equivariant (respectively,  $G$ -equivariant with respect to the adjoint action) perverse sheaves on  $G$ .

We recall now that the category  $\mathcal{C}_{B \times B}$  has a natural monoidal structure with tensor product given by the convolution, see e.g. [L4, Section 1]. The group  $B$  acts freely on  $G \times G$  as follows:  $b \cdot (x, y) = (xb, b^{-1}y)$ . Let  $G \times_B G$  denote the quotient of  $G \times G$  by this action and let  $\pi : G \times G \rightarrow G \times_B G$  be the natural projection. Clearly, the multiplication map  $m(x, y) = xy$  is a well defined map

$m : G \times_B G \rightarrow G$ . For  $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{C}_{B \times B}$  let  $\widetilde{\mathcal{F}_1 \boxtimes \mathcal{F}_2}$  be the unique complex of sheaves on  $G \times_B G$  such that  $\pi^* \widetilde{\mathcal{F}_1 \boxtimes \mathcal{F}_2} = \mathcal{F}_1 \boxtimes \mathcal{F}_2$ . By definition, the convolution of  $\mathcal{F}_1$  and  $\mathcal{F}_2$  is  $\mathcal{F}_1 * \mathcal{F}_2 := m_! \widetilde{\mathcal{F}_1 \boxtimes \mathcal{F}_2}$ . It follows from the Decomposition Theorem (see [BBD]) that  $\mathcal{F}_1 * \mathcal{F}_2 \in \mathcal{C}_{B \times B}$  and it is clear that the convolution is a bifunctor.

Let  $G \times_B G \times_B G$  be the quotient of  $G \times G \times G$  by the free  $B \times B$ -action via  $(b_1, b_2) \cdot (x, y, z) = (xb_1, b_1^{-1}yb_2, b_2^{-1}z)$  with obvious projection  $\pi_2 : G \times G \times G \rightarrow G \times_B G \times_B G$  and multiplication map  $m_2 : G \times_B G \times_B G \rightarrow G$ ,  $m_2(x, y, z) = xyz$ . For  $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \in \mathcal{C}_{B \times B}$  the convolutions  $(\mathcal{F}_1 * \mathcal{F}_2) * \mathcal{F}_3$  and  $\mathcal{F}_1 * (\mathcal{F}_2 * \mathcal{F}_3)$  are both canonically isomorphic to  $(m_2)_! (\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_3)$  where  $\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_3$  is the unique complex on  $G \times_B G \times_B G$  such that  $\pi_2^* (\mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_3) = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_3$ . Identifying  $(\mathcal{F}_1 * \mathcal{F}_2) * \mathcal{F}_3$  and  $\mathcal{F}_1 * (\mathcal{F}_2 * \mathcal{F}_3)$  via these isomorphisms we endow the category  $\mathcal{C}_{B \times B}$  with associativity constraint satisfying the pentagon axiom. Moreover,  $\mathcal{C}_{B \times B}$  is a monoidal category with unit object given by the constant sheaf on  $B \subset G$ .

**Remark 2.9.** The indecomposable objects of the category  $\mathcal{C}_{B \times B}$  are  $IC_w[i]$  where  $IC_w$  is the intersection cohomology complex (see [BBD]) of the Bruhat cell corresponding to  $w \in W$  and  $[i]$  stands for the shift. It is well known (see e.g. [L4, §1.4]) that the (split) Grothendieck ring of the category  $\mathcal{C}_{B \times B}$  identifies with *Hecke algebra*; under this identification the objects  $IC_w[i]$  correspond to the elements of Kazhdan-Lusztig basis multiplied by  $i$ -th power of the Hecke algebra parameter.

There is an equivariantization functor  $\Gamma : \mathcal{C}_{B \times B} \rightarrow \mathcal{C}_G$  defined as follows (see [MV, Section 1]). Let  $G \tilde{\times}_B G$  be the quotient of  $G \times G$  by the following free  $B$ -action:  $b \odot (g, x) = (b^{-1}g, b^{-1}xb)$ . We have a canonical projection  $\tilde{\pi} : G \times G \rightarrow G \tilde{\times}_B G$  and the adjoint action map  $a : G \tilde{\times}_B G \rightarrow G$ ,  $a(g, x) = g^{-1}xg$ . For  $\mathcal{F} \in \mathcal{C}_{B \times B}$  let  $\tilde{\mathcal{F}}$  be a unique complex on  $G \tilde{\times}_B G$  such that  $\tilde{\pi}^* \mathcal{F} = K \boxtimes \tilde{\mathcal{F}}$ . We set  $\Gamma(\mathcal{F}) := a_! (\tilde{\mathcal{F}})$ . It is clear that  $\Gamma(\mathcal{F})$  is  $G$ -equivariant complex on  $G$  (with respect to the adjoint action) and the Decomposition Theorem (see [BBD]) implies that  $\Gamma(\mathcal{F})$  is semisimple. In other words,  $\Gamma$  is a functor  $\mathcal{C}_{B \times B} \rightarrow \mathcal{C}_G$ .

**Remark 2.10.** We recall that irreducible constituents of perverse cohomology of complexes  $\Gamma(\mathcal{F})$  are by definition *character sheaves* (with trivial central character, or unipotent) on  $G$ , see [L3, Sections 2, 11] and [MV, Lemma 2.3].

We now define commutator structure on the functor  $\Gamma$ . Let  $G \tilde{\times}_B (G \times_B G)$  be the quotient of  $G \times G \times G$  with respect to the following free  $B \times B$ -action:  $(b_1, b_2) \cdot (g, x, y) = (b_1^{-1}g, b_1^{-1}xb_2, b_2^{-1}yb_1)$ . We have an obvious projection  $\tilde{\pi}_2 : G \times G \times G \rightarrow G \tilde{\times}_B (G \times_B G)$  and a well defined map  $a_2 : G \tilde{\times}_B (G \times_B G) \rightarrow G$ ,  $a_2(g, x, y) = g^{-1}xyg$ . We have an obvious isomorphism  $\Gamma(\mathcal{F}_1 * \mathcal{F}_2) = (a_2)_! (\overline{\mathcal{F}_1 \boxtimes \mathcal{F}_2})$  where  $\overline{\mathcal{F}_1 \boxtimes \mathcal{F}_2}$  is a unique complex on  $G \tilde{\times}_B (G \times_B G)$  such that  $\tilde{\pi}_2^* (\mathcal{F}_1 \boxtimes \mathcal{F}_2) = K \boxtimes \mathcal{F}_1 \boxtimes \mathcal{F}_2$ .

Consider the following (well defined) map  $\rho : G \tilde{\times}_B (G \times_B G) \rightarrow G \tilde{\times}_B (G \times_B G)$ ,  $\rho(g, x, y) = (yg, y, x)$ . For  $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{C}_{B \times B}$  there is a unique isomorphism  $\tilde{u}_{\mathcal{F}_1, \mathcal{F}_2} : \rho_! (\overline{\mathcal{F}_1 \boxtimes \mathcal{F}_2}) \simeq \overline{\mathcal{F}_2 \boxtimes \mathcal{F}_1}$  inducing the identity map in every stalk (clearly, the stalks of both  $\rho_! (\overline{\mathcal{F}_1 \boxtimes \mathcal{F}_2})$  and  $\overline{\mathcal{F}_2 \boxtimes \mathcal{F}_1}$  at  $(g, x, y) \in G \tilde{\times}_B (G \times_B G)$  are canonically isomorphic to  $(\mathcal{F}_2)_x \otimes (\mathcal{F}_1)_y$ ). Now observe that  $a_2 \circ \rho = a_2$ . Thus we have an isomorphism  $u_{\mathcal{F}_1, \mathcal{F}_2} : \Gamma(\mathcal{F}_1 * \mathcal{F}_2) \simeq \Gamma(\mathcal{F}_2 * \mathcal{F}_1)$  defined as composition

$$\Gamma(\mathcal{F}_1 * \mathcal{F}_2) = (a_2)_! (\overline{\mathcal{F}_1 \boxtimes \mathcal{F}_2}) = (a_2 \circ \rho)_! (\overline{\mathcal{F}_1 \boxtimes \mathcal{F}_2}) \xrightarrow{(a_2)_! (\tilde{u}_{\mathcal{F}_1, \mathcal{F}_2})} (a_2)_! (\overline{\mathcal{F}_2 \boxtimes \mathcal{F}_1}) = \Gamma(\mathcal{F}_2 * \mathcal{F}_1).$$

The following result is straightforward:

**Proposition 2.11.** *The functor  $\Gamma$  together with isomorphism  $u_{\bullet,\bullet}$  is a commutator functor  $\mathcal{C}_{B \times B} \rightarrow \mathcal{C}_G$ .  $\square$*

There is a canonical automorphism  $\Theta$  of the identity functor of the category  $\mathcal{C}_G$  defined as follows (see [O2, Definition 2.1]): for any  $\mathcal{F} \in \mathcal{C}_G$  by definition of equivariance there is an isomorphism  $ad^*(\mathcal{F}) \simeq p^*(\mathcal{F})$  where  $p : G \times G \rightarrow G$  is the second projection and  $ad : G \times G \rightarrow G$  is the adjoint action map,  $ad(g, x) = g^{-1}xg$ ; taking pullback of this isomorphism with respect to the diagonal map  $\Delta : G \rightarrow G \times G$  and noticing that  $\Delta \circ ad = \Delta \circ p = \text{Id}$  we get an automorphism  $\Theta_{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{F}$ . In other words, the automorphism  $\Theta_{\mathcal{F}}$  at the stalk  $\mathcal{F}_x$  is precisely  $ad(x) : \mathcal{F}_x \simeq \mathcal{F}_{x^{-1}xx}$ . We have the following

**Proposition 2.12.** *The canonical automorphism of the commutator functor  $\Gamma$  equals  $\Theta$ , that is  $u_{1,\mathcal{F}} = \Theta_{\Gamma(\mathcal{F})}$ .  $\square$*

**Remark 2.13.** Our construction of commutator structure on the functor  $\Gamma$  is just a more explicit version of construction in [BFO1, §6] (we will not need this fact in what follows). The advantage of the present version is that it makes Proposition 2.12 easy.

**2.3. Truncated convolution and equivariantization.** We recall (see e.g. [L1, Chapter 5]) that the Weyl group  $W$  is partitioned into two sided cells,  $W = \sqcup \mathbf{c}$ . Remind that there is a partial order  $\leq_{LR}$  on the set of two sided cells. Let  $a(\mathbf{c})$  denote the common value of Lusztig's  $a$ -function on  $w \in W$ , see [L4, §2.3].

Let  $\mathcal{P}_{B \times B}$  denote the full subcategory of  $\mathcal{C}_{B \times B}$  consisting of perverse sheaves. Let  $\mathcal{P}_{B \times B}^{\mathbf{c}}$  denote the full subcategory of  $\mathcal{P}_{B \times B}$  consisting of direct sums of perverse sheaves  $IC_w$ ,  $w \in \mathbf{c}$  and let  $pr^{\mathbf{c}} : \mathcal{P}_{B \times B} \rightarrow \mathcal{P}_{B \times B}^{\mathbf{c}}$  be the obvious projection functor. Let  ${}^pH$  denote the perverse cohomology functor. Consider the following bifunctor  $\mathcal{P}_{B \times B}^{\mathbf{c}} \times \mathcal{P}_{B \times B}^{\mathbf{c}} \rightarrow \mathcal{P}_{B \times B}^{\mathbf{c}}$ :

$$\mathcal{F}_1 \circledast \mathcal{F}_2 := pr^{\mathbf{c}}({}^pH^{a(\mathbf{c})}(\mathcal{F}_1 * \mathcal{F}_2)).$$

It is explained in [L4, Section 2] that the associativity constraint of the convolution category  $\mathcal{C}_{B \times B}$  restricts to the associativity constraint for the bifunctor  $\circledast$ . Moreover, there exists the unit object  $\mathbf{1}_{\mathbf{c}} \in \mathcal{P}_{B \times B}^{\mathbf{c}}$  (see [L4, §2.9]), so the category  $\mathcal{P}_{B \times B}^{\mathbf{c}}$  has a monoidal structure. We have the following result

**Proposition 2.14.** ([BFO1, p. 222]) *The monoidal category  $\mathcal{P}_{B \times B}^{\mathbf{c}}$  is rigid. Hence  $\mathcal{P}_{B \times B}^{\mathbf{c}}$  is a multi-fusion category in a sense of [ENO, §2.4].  $\square$*

**Remark 2.15.** The Grothendieck ring of the category  $\mathcal{P}_{B \times B}^{\mathbf{c}}$  identifies with Lusztig's asymptotic Hecke ring  $J_{\mathbf{c}}$ , see [L4, §2.6]. This together with [L1, Corollary 12.16] implies that the multi-fusion category  $\mathcal{P}_{B \times B}^{\mathbf{c}}$  is indecomposable in the sense of [ENO, §2.4].

Let  $\mathcal{P}_G$  be the full subcategory of  $\mathcal{C}_G$  consisting of direct sums of (unshifted) unipotent character sheaves, see Remark 2.10. It is known (see [L3, Section 16], [G, §3.4]) that there is a unique direct sum decomposition (or, equivalently, partition of the set of isomorphism classes of unipotent character sheaves)  $\mathcal{P}_G = \bigoplus_{\mathbf{c}} \mathcal{P}_{\mathbf{c}}^{\mathbf{c}}$  with the following properties:

- (i) For  $w \in \mathbf{c}$  and  $i \in \mathbb{Z}$  we have  ${}^pH^i(\Gamma(IC_w)) \in \bigoplus_{\mathbf{c}' \leq_{LR} \mathbf{c}} \mathcal{P}_{\mathbf{c}'}^{\mathbf{c}'}$ ; moreover  ${}^pH^i(\Gamma(IC_w)) \in \bigoplus_{\mathbf{c}' <_{LR} \mathbf{c}} \mathcal{P}_{\mathbf{c}'}^{\mathbf{c}'}$  if  $|i| > a(\mathbf{c})$ .

(ii) For a simple perverse sheaf  $A \in \mathcal{P}_G^c$  there exists  $w \in \mathbf{c}$  such that  $A$  is a direct summand of  ${}^p H^{a(\mathbf{c})}(\Gamma(IC_w))$ .

Let  $\tilde{p}r^c$  denote the projection functor  $\mathcal{P}_G \rightarrow \mathcal{P}_G^c$ . We consider the functor  $\Gamma^c : \mathcal{P}_{B \times B}^c \rightarrow \mathcal{P}_G^c$ ,  $\Gamma^c(\mathcal{F}) := \tilde{p}r^c({}^p H^{a(\mathbf{c})}(\Gamma(\mathcal{F}))$ . Notice that the property (ii) implies that the functor  $\Gamma^c$  is surjective, that is each simple object of  $\mathcal{P}_G^c$  appears as a direct summand of some  $\Gamma^c(\mathcal{F})$ . The properties above imply that the restriction of the commutator structure of the functor  $\Gamma$  gives a well defined commutator structure  $u^c$  of the functor  $\Gamma^c$ , see [BFO1, §6]. We have

**Proposition 2.16.** *The functor  $\Gamma^c : \mathcal{P}_{B \times B}^c \rightarrow \mathcal{P}_G^c$  is a surjective commutator functor with canonical automorphism  $\Theta$ .*

*Proof.* We need only check the statement about the canonical automorphism of  $\Gamma^c$ . By definition the object  $\Gamma^c(\mathbf{1}_c \otimes \mathcal{F})$  is a direct summand of  $\Gamma(\mathbf{1}_c * \mathcal{F})$ . The canonical automorphism of  $\Gamma^c(\mathcal{F}) = \Gamma^c(\mathbf{1}_c \otimes \mathcal{F})$  equals the restriction of automorphism  $u_{\mathcal{F}, \mathbf{1}_c} \circ u_{\mathbf{1}_c, \mathcal{F}}$  of  $\Gamma(\mathbf{1}_c * \mathcal{F})$  to this direct summand. By Remark 2.4 the composition  $u_{\mathcal{F}, \mathbf{1}_c} \circ u_{\mathbf{1}_c, \mathcal{F}}$  equals the canonical automorphism of  $\Gamma(\mathbf{1}_c * \mathcal{F})$ . Now the result follows from Proposition 2.12.  $\square$

**2.4. Proof of Theorem 1.1.** Recall that  $\mathbf{c}$  is an exceptional two sided cell. Let  $\mathcal{P}^c = \mathcal{P}_{B \times B}^c$ .

The following result follows easily from [Sh2, p. 347] (for  $W$  of type  $E_7$ ) and [Sh1, Proposition 3.6] (for  $W$  of type  $E_8$ ):

**Proposition 2.17.** *Assume that  $\mathbf{c}$  is an exceptional two sided cell. The category  $\mathcal{P}_G^c$  has 4 isomorphism classes of simple objects. The automorphism  $\Theta$  acts as  $\text{Id}$  on two of them and as  $\pm\sqrt{-1}\text{Id}$  on two others.  $\square$*

It follows from Proposition 2.16 that the canonical automorphism of the commutator functor  $\Gamma^c : \mathcal{P}^c \rightarrow \mathcal{P}_G^c$  has order 4.

Let  $\mathbf{1}_c = \oplus_i \mathbf{1}_i$  be the decomposition of the unit object  $\mathbf{1}_c \in \mathcal{P}^c$  into irreducible summands. It is clear that  $\{\mathbf{1}_i\}_i$  are orthogonal idempotents, that is  $\mathbf{1}_i \otimes \mathbf{1}_i \simeq \mathbf{1}_i$  and  $\mathbf{1}_i \otimes \mathbf{1}_j = 0$  if  $i \neq j$ . For any simple object  $X \in \mathcal{P}^c$  there are unique  $i$  and  $j$  such that  $X = \mathbf{1}_i \otimes X \otimes \mathbf{1}_j$ ; observe that  $i \neq j$  implies  $\Gamma^c(X) = 0$  (indeed, in this case  $\Gamma^c(X) = \Gamma^c(\mathbf{1}_i \otimes X) = \Gamma^c(X \otimes \mathbf{1}_j) = \Gamma^c(0) = 0$ ). Thus there exists  $i$  and a simple object  $X = \mathbf{1}_i \otimes X \otimes \mathbf{1}_i \in \mathcal{P}^c$  such that  $\Gamma^c(X)$  contains an irreducible summand  $\mathcal{F}$  with  $\Theta_{\mathcal{F}} = \pm\sqrt{-1}\text{Id}_{\mathcal{F}}$ . We fix such  $i$ .

Clearly  $\mathbf{1}_i \otimes \mathcal{P}^c \otimes \mathbf{1}_i \subset \mathcal{P}^c$  is a tensor subcategory with unit object  $\mathbf{1}_i$  and the restriction  $\Gamma^c|_{\mathbf{1}_i \otimes \mathcal{P}^c \otimes \mathbf{1}_i}$  is a commutator functor. The choice of  $i$  guarantees that the order of the canonical automorphism of  $\Gamma^c|_{\mathbf{1}_i \otimes \mathcal{P}^c \otimes \mathbf{1}_i}$  is 4. The Grothendieck ring of the category  $\mathbf{1}_i \otimes \mathcal{P}^c \otimes \mathbf{1}_i$  is computed in [L2, Corollary 3.12] (cf. Remark 2.15); we see that this category has precisely 2 isomorphism classes of simple objects  $\mathbf{1}_i$  and  $\delta_i$ , furthermore  $\delta_i \otimes \delta_i \simeq \mathbf{1}_i$ . The following result is well known; it is a special case of results of [S]:

**Lemma 2.18.** *Let  $\mathcal{C}$  be a fusion category with two (isomorphism classes of) simple objects: the unit object  $\mathbf{1}$  and one more object  $\delta$  such that  $\delta \otimes \delta \simeq \mathbf{1}$ . Then either  $\mathcal{C} \simeq \text{Vec}_{\mathbb{Z}/2\mathbb{Z}}$  or  $\mathcal{C} \simeq \text{Vec}_{\mathbb{Z}/2\mathbb{Z}}^\omega$ .  $\square$*

In view of Example 2.8 we see that  $\mathbf{1}_i \otimes \mathcal{P}^c \otimes \mathbf{1}_i$  can not be equivalent to  $\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}$  (notice that the commutator functor  $\Gamma^c|_{\mathbf{1}_i \otimes \mathcal{P}^c \otimes \mathbf{1}_i}$  has the left adjoint since it is a functor between semisimple categories). Hence  $\mathbf{1}_i \otimes \mathcal{P}^c \otimes \mathbf{1}_i \simeq \text{Vec}_{\mathbb{Z}/2\mathbb{Z}}^\omega$ .

Let  $\mathcal{C}$  be a multi-fusion category and let  $e \in \mathcal{C}$  be a direct summand of the unit object. Then  $e \otimes \mathcal{C} \otimes e$  is a tensor subcategory of  $\mathcal{C}$  and  $e \otimes \mathcal{C}$  is a *module category* over  $e \otimes \mathcal{C} \otimes e$  via the left multiplication, see [O1]. Each object  $X \in \mathcal{C}$  gives rise to a functor  $? \otimes X : e \otimes \mathcal{C} \rightarrow e \otimes \mathcal{C}$  commuting with the module structure above. Thus we get a tensor functor from  $\mathcal{C}$  to the category  $\text{Fun}_{e \otimes \mathcal{C} \otimes e}(e \otimes \mathcal{C}, e \otimes \mathcal{C})$  of  $e \otimes \mathcal{C} \otimes e$ -module endofunctors of  $e \otimes \mathcal{C}$ . The following result is well known:

**Lemma 2.19.** *Assume that multi-fusion category  $\mathcal{C}$  is indecomposable. The functor above is an equivalence  $\mathcal{C} \simeq \text{Fun}_{e \otimes \mathcal{C} \otimes e}(e \otimes \mathcal{C}, e \otimes \mathcal{C})$ .*

*Proof.* The category  $e \otimes \mathcal{C}$  is a right module category over  $\mathcal{C}$  and the obvious functor  $e \otimes \mathcal{C} \otimes e \rightarrow \text{Fun}_{\mathcal{C}}(e \otimes \mathcal{C}, e \otimes \mathcal{C})$  is an equivalence with the inverse functor  $F \mapsto F(e)$ . In the language of [O1, §4.2], this means that the category  $e \otimes \mathcal{C} \otimes e$  is dual to the category  $\mathcal{C}$ . Now, the result follows by duality, see [O1, Corollary 4.1].  $\square$

Thus we can reconstruct multi-fusion category  $\mathcal{C}$  from smaller category  $e \otimes \mathcal{C} \otimes e$  and module category  $e \otimes \mathcal{C}$ . We apply this to the case  $\mathcal{C} = \mathcal{P}^{\mathbf{c}}$  and  $e = \mathbf{1}_i$ . It is well known (see e.g. [O1, Remark 6.2(iii)]) that the fusion category  $\mathbf{1}_i \otimes \mathcal{P}^{\mathbf{c}} \otimes \mathbf{1}_i \simeq \text{Vec}_{\mathbb{Z}/2\mathbb{Z}}^{\omega}$  has only one indecomposable semisimple module category, namely the regular module category  $\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}^{\omega}$ . Hence  $\mathcal{P}^{\mathbf{c}} \simeq \text{Fun}_{\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}^{\omega}}(\mathcal{M}, \mathcal{M})$  where  $\mathcal{M}$  is a direct sum of several copies of this module category (the number of indecomposable summands in  $\mathcal{M}$  equals the number of irreducible summands in  $\mathbf{1}_{\mathbf{c}}$ ). The same argument applies to the multi-fusion category  $\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}^{\omega} \boxtimes \text{Coh}(Y' \times Y')$  (in this case the irreducible summands of the unit object are naturally labeled by the elements of the set  $Y'$ ). Thus choosing the set  $Y'$  of appropriate size we get

$$\mathcal{P}^{\mathbf{c}} \simeq \text{Fun}_{\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}^{\omega}}(\mathcal{M}, \mathcal{M}) \simeq \text{Vec}_{\mathbb{Z}/2\mathbb{Z}}^{\omega} \boxtimes \text{Coh}(Y' \times Y')$$

and Theorem 1.1 is proved.

**Remark 2.20.** In [BFO2, Section 4] the category  $\mathcal{P}_G^{\mathbf{c}}$  is endowed with tensor structure; moreover [BFO2, Theorem 5.3] states that  $\mathcal{P}_G^{\mathbf{c}} \simeq \mathcal{Z}(\mathcal{P}^{\mathbf{c}})$  and the functor  $\Gamma^{\mathbf{c}}$  is isomorphic to the induction functor  $\text{Ind} : \mathcal{P}^{\mathbf{c}} \rightarrow \mathcal{Z}(\mathcal{P}^{\mathbf{c}}) \simeq \mathcal{P}_G^{\mathbf{c}}$  (one can use it to give a shorter proof of Theorem 1.1). This implies that for an exceptional cell  $\mathbf{c}$  we have  $\mathcal{P}^{\mathbf{c}} \simeq \mathcal{Z}(\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}^{\omega})$  (see [BFO2, Corollary 5.4(b)] for the case when  $\mathbf{c}$  is not exceptional).

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